1. [10 pts.]. A ladder of length \( L \) and mass \( M \) has its bottom end attached to the ground by a pivot. It makes an angle \( \theta \) with the horizontal, and is held up by a massless stick of length \( \ell \) which is also attached to the ground by a pivot (see Figure). The ladder and the stick are perpendicular to each other. Find the force that the stick exerts on the ladder.

Consider rotational equilibrium of the ladder about the pivot. The torque due to the weight is given by

\[
\tau_{Mg} = -\frac{1}{2} MgL \cos \theta
\]

and the torque due to the force \( F \) exerted by the stick is

\[
\tau_F = +\frac{F\ell}{\tan \theta}
\]

Since the net torque is zero,

\[
\tau_{net} = \tau_{Mg} + \tau_F = \frac{F\ell}{\tan \theta} - \frac{1}{2} MgL \cos \theta = 0
\]

Solving this equation for \( F \), we obtain

\[
F = \frac{MgL \cos \theta \tan \theta}{2\ell} = \frac{MgL \sin \theta}{2\ell}
\]

ANS: \( F = \frac{MgL \sin \theta}{2\ell} \)
2. [10 pts.] Consider a particle of mass $m$ moving in a two-dimensional central force field $\vec{F} = \lambda \vec{r}$, where $\vec{r} = xe_x + ye_y$ and $\lambda$ is a positive constant. At an initial time $t = 0$, its position is $\vec{r}_0 = ae_x + be_y$ where $a$ and $b$ are positive constants. The initial velocity is not known. However, the product of velocity components does not depend on time and equals to a non-zero constant all time. (That is $v_x(t)v_y(t) = \text{non-zero constant}$). Show that the particle moves in a hyperbola (that means $x(t)y(t) = \text{constant}$.)

The Newton equations for each component are

$$m \ddot{x} = \lambda x, \quad m \ddot{y} = \lambda y$$

which can be integrated and we obtain the general solutions

$$x(t) = Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}$$
$$y(t) = Ce^{\sqrt{\lambda}t} + De^{-\sqrt{\lambda}t}$$

and its velocity components

$$\dot{x}(t) = \sqrt{\lambda} \left( Ae^{\sqrt{\lambda}t} - Be^{-\sqrt{\lambda}t} \right)$$
$$\dot{y}(t) = \sqrt{\lambda} \left( Ce^{\sqrt{\lambda}t} - De^{-\sqrt{\lambda}t} \right)$$

Now we apply the given conditions. First, we consider the initial conditions, $x(0) = a$ and $y(0) = b$ and find

$$A + B = a \quad \text{and} \quad C + D = b$$

Next, we apply the condition for the velocity components:

$$\dot{x}\dot{y} = \lambda \left( ACe^{2\sqrt{\lambda}t} + BDe^{-2\sqrt{\lambda}t} - (AD + BC) \right) = \text{const}$$

Therefore, we find

$$AC = 0, \quad BD = 0$$

By solving Eqs. (10) and (11), we find two solutions

$$[A = 0, B = a, C = b, D = 0] \quad \text{and} \quad [A = a, B = 0, C = 0, D = b]$$

In either case, $AD + BC = ab$. Under this condition, the product of the position components is

$$xy = ACe^{2\sqrt{\lambda}t} + BDe^{-2\sqrt{\lambda}t} + AD + BC = AD + BC = ab$$

Hence, the trajectory is hyperbolic. ■
3. [20 pts.] A uniform coin of mass $M$ and radius $R$ stands vertically on the right end of a horizontal uniform plank of mass $M$ and length $L$, as shown in Figure. The plank is pulled to the right with a constant force $F$. Assume that the coin does not slip with respect to the plank. What are the accelerations of the plank and coin?

Newton equation for the plank is given by

$$F - f = Ma_p$$

(14)

where $f$ is the friction exerted on the coin by the plank and $a_p$ the acceleration of the plank. For the translational motion of the coin, we have

$$f = Ma_c$$

(15)

where $a_c$ is the acceleration of the coin. Using torque $\tau = fR$ and moment of inertia $I = MR^2/2$ (see below), the equation of rotational motion is given by

$$fR = \left(\frac{MR^2}{2}\right) \alpha$$

(16)

where $\alpha$ is the rotational acceleration. Since the coin does not slip, three accelerations are not independent and their relation is

$$\alpha R = a_p - a_c$$

(17)

Solving Eqs. (14)-(17) we can obtain all accelerations.

ANS: $a_c = \frac{F}{4M}$ and $a_p = \frac{3F}{4M}$

Calculation of moment of inertia

$$I = \int r^2 dm = \int_0^R \int_0^{2\pi} r^2 \frac{M}{\pi R^2} r \, dr \, d\theta$$

$$= \frac{M}{\pi R^2} \int_0^R \int_0^{2\pi} r^3 \, dr \, d\theta = \frac{1}{2} MR^2$$
4. [20 pts.] Two springs have spring constants, \(k_1\) and \(k_2\), respectively and one end of each spring is attached to a separate wall as shown in Figure. A ball of mass \(m\) connects the two springs. The ball can oscillate only horizontally. A massless rigid rod of length \(L\) is attached to the ball and is free to rotate around the ball (that is, the angle \(\theta\) can vary from \(-\pi\) to \(+\pi\)). Another ball of mass \(M\) is attached to the other end of the rod. The position \(x\) of \(m\) is measured from the equilibrium position of the springs and the coordinate \(\theta\) is measured from the vertical.

(a) Find the Lagrangian of the system and the Euler-Lagrange equations for each coordinate.

(b) When \(m\) is negligibly small compared to \(M\) and the amplitude of the oscillation is small, show that 
\[
\theta \approx \frac{k_1 + k_2}{Mg} x.
\]

(c) Using the above approximation, find the frequency of the small amplitude oscillation.

\(\text{(a)}\) The kinetic energy of mass \(m\) is 
\[
T_m = \frac{m}{2} \dot{x}^2
\]
Noting that the Cartesian coordinates of mass \(M\) are \(X = x + \ell \sin \theta\) and \(Y = -\ell \cos \theta\), its kinetic energy is 
\[
T_M = \frac{M}{2} \left( \dot{X}^2 + \dot{Y}^2 \right) = \frac{M}{2} \left( \dot{x}^2 + 2\ell \dot{x} \dot{\theta} \cos \theta + \ell^2 \dot{\theta}^2 \right)
\]
The potential energy stored in the springs is given by 
\[
U_k = \frac{k_1 + k_2}{2} x^2.
\]
The potential energy of mass \(M\) is 
\[
U_M = MgY = -Mg\ell \cos \theta
\]
Combining all these energy, we obtain Lagrangian 
\[
\mathcal{L} = T_m + T_M - U_k - U_M = \frac{m}{2} \dot{x}^2 + \frac{M}{2} \left( \dot{x}^2 + 2\ell \dot{x} \dot{\theta} \cos \theta + \ell^2 \dot{\theta}^2 \right) - \frac{k_1 + k_2}{2} x^2 + Mg\ell \cos \theta
\]
The corresponding Euler-Lagrange equations are 
\[
\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = -(k_1 + k_2)x - (m + M)\ddot{x} - M\ell \cos \theta \ddot{\theta} + ML \sin \theta \ddot{\theta}^2 = 0
\]
\[
\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = -Mg\ell \sin \theta - M\ell^2 \ddot{\theta} - M\ell \cos \theta \ddot{x} = 0
\]
\(\text{(b)}\) Assuming \(m \ll M\), \(\sin \theta \approx \theta\) and \(\cos \theta \approx 1\), and neglecting \(\dot{\theta}^2\) or higher orders, the equations of motion become 
\[
-(k_1 + k_2)x - M\ddot{x} - M\ell \ddot{\theta} = 0
\]
\[
Mg\theta + M\ell \ddot{\theta} + M\ddot{x} = 0
\]
By adding these two equations, we obtain 
\[
Mg\theta - (k_1 + k_2)x = 0
\]
Hence, \(\theta \approx \frac{k_1 + k_2}{Mg} x\).
(c) Under the small amplitude approximation, $\theta$ and $x$ are linearly related as shown in part (b). Using Eq. (27), we eliminate $\theta$ in Eq. (25). The resulting equation of motion for $x$ is given by

$$-(k_1 + k_2)x - M\ddot{x} - \frac{f(k_1 + k_2)}{g}\ddot{x} = 0$$

(28)

Rearranging it, we obtain a simple harmonic oscillator

$$\ddot{x} = -\frac{(k_1 + k_2)g}{Mg + (k_1 + k_2)\ell} x$$

(29)

Hence, the frequency of the small amplitude oscillation is

$$\omega = \sqrt{\frac{(k_1 + k_2)g}{Mg + (k_1 + k_2)\ell}}.$$

Comment: If the spring constants are very large, $x$ does not change. Then, the system is a simple pendulum. Taking the limit $k_1 + k_2 \to \infty$ the frequency indeed approaches the well known value $\omega \to \sqrt{g/\ell}$. 
5. [20 pts.] A particle of mass $m$ and charge $q$ moves under the influence of uniform electric field $\vec{E} = E\hat{e}_y$ and magnetic field $\vec{B} = B\hat{e}_z$, which can be obtained from a scalar potential $\phi = -Ey$ and a vector potential $\vec{A} = \frac{B}{2}(-y\hat{e}_x + x\hat{e}_y)$, through $\vec{E} = -\nabla\phi$ and $\vec{B} = \nabla \times \vec{A}$.

(a) Find the equations of motion using either Lagrangian or Hamiltonian method.

(b) Assuming that the particle is initially at rest at the coordinate origin, find the trajectory of the particle.

Lagrangian Method:

(a) The Lagrangian of a charged particle is given in general by $L = \frac{m}{2} \dot{\vec{r}}^2 - q(\phi - \vec{r} \cdot \vec{A})$. For the present problem, the Lagrangian is

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + qEy - qB \frac{\dot{y}}{2} x + qB \frac{\dot{y}}{2} x$$

(30)

The corresponding Euler-Lagrange equations are

$$m\ddot{x} = qBy$$

(31)

$$m\ddot{y} = qE - qB \dot{x}$$

(32)

$$m\ddot{z} = 0$$

(33)

(b) Obviously, $z(t) = 0$ all time. Integrating Eq. (31) once with respect to time

$$m\dot{x} = qBy$$

(34)

where we applied the given initial conditions. Substituting this result into Eq. (32), we obtain an ODE:

$$\ddot{y} = -\left( \frac{qB}{m} \right)^2 \left( y - \frac{qE}{qB^2} \right)$$

(35)

Introducing a new variable $Y = y - \frac{qE}{qB^2}$, $y$ satisfies the equation of a simple harmonic oscillator

$$\ddot{Y} = -\left( \frac{qB}{m} \right)^2 Y$$

(36)

Hence

$$y(t) = a \sin(\omega t) + b \cos(\omega t) + \frac{qE}{qB^2}$$

(37)

where $\omega = \frac{qB}{m}$. Applying the initial conditions ($y(0) = 0$ and $\dot{y}(0) = 0$), we obtain $a = 0$ and $b = -\frac{qE}{qB^2}$. The final expression is

$$y(t) = \frac{qE}{qB^2} [1 - \cos(\omega t)]$$

(38)

Substituting $y$ into Eq. (31),

$$\ddot{x} = \frac{qE}{m} \sin \omega t$$

(39)

Integrating it, we obtain

$$x = -\frac{qE}{m\omega^2} \sin \omega t + ct + d$$

(40)

Applying the initial conditions ($x(0) = 0$ and $\dot{x}(0) = 0$), we obtain $c = \frac{qE}{m\omega^2}$ and $d = 0$. Hence,

$$x(t) = \frac{qE}{qB^2} \sin \omega t + \frac{E}{B} t$$

(41)
Hamiltonian Method:
(a) The Hamiltonian of a charged particle is given in general by
\[ H = \frac{m}{2} \left( \dot{p} - q \dot{A} \right)^2 + q\phi \] (42)
For the present problem, the Hamiltonian is
\[ H = \frac{m}{2} \left( p_x + \frac{qB}{2} y \right)^2 + \frac{m}{2} \left( p_y - \frac{qB}{2} x \right)^2 + \frac{m}{2} p_z^2 \] (43)
The corresponding Canonical equations of motion are
\[ \dot{x} = \frac{\partial H}{\partial p_x} \rightarrow \dot{x} = \frac{1}{m} \left( p_x + \frac{qB}{2} y \right) \] (44)
\[ \dot{p}_x = -\frac{\partial H}{\partial x} \rightarrow \dot{p}_x = \frac{qB}{2m} \left( p_y - \frac{qB}{2} x \right) \] (45)
\[ \dot{y} = \frac{\partial H}{\partial p_x} \rightarrow \dot{y} = \frac{1}{m} \left( p_y - \frac{qB}{2} x \right) \] (46)
\[ \dot{p}_y = -\frac{\partial H}{\partial y} \rightarrow \dot{p}_y = -\frac{qB}{2m} \left( p_x + \frac{qB}{2} y \right) - qE \] (47)
\[ \dot{z} = \frac{\partial H}{\partial p_x} \rightarrow \dot{z} = \frac{p_z}{m} \] (48)
\[ \dot{p}_z = -\frac{\partial H}{\partial z} \rightarrow \dot{p}_z = 0 \] (49)
(b) Taking time derivative of Eq. (44),
\[ \ddot{x} = \frac{1}{m} \left( \dot{p}_x + \frac{qB}{2} \dot{y} \right) \] (50)
Using Eqs. (45) and (46), we obtain
\[ \dot{p}_x = \frac{qB}{2} \dot{y} \] (51)
Substituting this into Eq. (50),
\[ \ddot{x} = \frac{qB}{m} \dot{y} \] (52)
Similarly, taking time derivative of Eq. (46),
\[ \ddot{y} = \frac{1}{m} \left( \dot{p}_y - \frac{qB}{2} \dot{x} \right) \] (53)
Using Eqs. (44) and (47), we obtain
\[ \dot{p}_y = \frac{qB}{2} \dot{x} - qE \] (54)
Substituting this into Eq. (53),
\[ \ddot{y} = -\frac{qB}{m} \dot{x} - \frac{qE}{m} \] (55)
For \( z \) component, use Eqs. (48) and (49) and we find
\[ \ddot{z} = 0 \] (56)
Equations (52), (54), and (56) equal to the Euler-Lagrange equations (31)-(33). See the solution for the Lagrangian method for the rest of calculation.
6. [20 pts.] A planet of mass $m$ moves around a star of mass $M$. They are attracted to each other by gravitational potential $U = -\frac{GmM}{r}$. The distance from the start to the planet oscillates between $r_{\text{min}}$ and $r_{\text{max}}$. Show that the energy $E$ and the angular momentum $L$ of the system are given by

$$E = -\frac{GmM}{r_{\text{min}} + r_{\text{max}}}, \quad L = \sqrt{2\mu GM \frac{r_{\text{min}}r_{\text{max}}}{r_{\text{min}} + r_{\text{max}}}}$$

where $\mu$ is the reduced mass.

Using the energy conservation principle for the relative motion, we have

$$\frac{\mu}{2} \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{GmM}{r} = E \quad (57)$$

where $\mu = \frac{mM}{m+M}$. At the turning points, $r_{\text{min}}$ and $r_{\text{max}}$, the radial kinetic energy is zero. Hence, $r_{\text{min}}$ and $r_{\text{max}}$ are two roots of

$$\frac{L^2}{2\mu r^2} - \frac{GmM}{r} = E \quad (58)$$

Writing this equation in a standard quadratic equation:

$$r^2 + \frac{GmM}{E} r - \frac{L^2}{2\mu E} = 0 \quad (59)$$

and using the properties of roots, we find the following relations:

$$r_{\text{min}} + r_{\text{max}} = -\frac{GmM}{E} \quad (60)$$

$$r_{\text{min}} r_{\text{max}} = -\frac{L^2}{2\mu E} \quad (61)$$

From Eq (60), we obtain the energy

$$E = -\frac{GmM}{r_{\text{min}} + r_{\text{max}}} \quad (62)$$

Substituting this $E$ into Eq (61) and solve it for $L$, we obtain the angular momentum:

$$L = \sqrt{-r_{\text{min}}r_{\text{max}}2\mu E} = \sqrt{2\mu GM \frac{r_{\text{min}}r_{\text{max}}}{r_{\text{min}} + r_{\text{max}}}} \quad (63)$$